Anthony Treadwell

Finishing the Computation from Last class 12/1/21

ex: Flux of $\vec{v} = \langle z, y, x \rangle$ across unit sphere at origin.

$$\iint_S \vec{v} \cdot d\vec{s} = \iint_{D_1} 0 \, d\vec{s} + \iint_{D_2} \sin^3(\phi)\sin^2(\theta)\, dA$$

$$= \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \sin^3(\phi)\sin^2(\theta)\, d\theta\, d\phi$$

$$= \int_{\phi=0}^{\pi} \sin^3(\phi) \frac{1}{2} \int_{\theta=0}^{2\pi} (1-\cos(2\theta))\, d\theta\, d\phi$$

$$= \frac{1}{2} \int_{\phi=0}^{\pi} \sin(\phi)(1-\cos^2(\phi)) \left[\theta - \frac{1}{2}\sin(2\theta)\right]_{\theta=0}^{2\pi} d\phi$$

$$= \frac{1}{2}(2\pi - 0 - 0) \int_{\phi=0}^{\pi} -(1-u^2)\, du \qquad \begin{aligned} u &= \cos(\phi) \\ du &= -\sin\phi\, d\phi \end{aligned}$$

$$= -\pi \left[u - \frac{1}{3}u^3\right]_{\phi=0}^{\pi}$$

$$= -\pi \left[\cos(\phi) - \frac{1}{3}\cos^3(\phi)\right]_{\phi=0}^{\pi} = -\pi\left((-1+\tfrac{1}{3}) - (1 - \tfrac{1}{3})\right)$$

$$\boxed{= \frac{4\pi}{3}} \checkmark$$

Ex: Compute the flux of $\vec{F} = \langle y, x, z \rangle$ on boundry of Solid enclosed by paraboloid $z = 1 - x^2 - y^2$ and plane $z = 0$

Picture

Solution: our computation breaks up over the two pieces in our picture (i.e. $S = S_1 \cup S_2$)

Parameterization: S_1: $\vec{S}(u,v) = \langle u\cos(v), u\sin(v), 1 - u^2 \rangle$

(r, θ)

$D = [0,1] \times [0, 2\pi]$

$$\vec{F}(\vec{S}(u,v)) = \langle u\sin(v), u\cos(v), 1 - u^2 \rangle$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(S(u,v)) \cdot (S_u \times S_v)\, dA$$

$$\vec{S}_u = \langle \cos(v), \sin(v), -2u \rangle$$

$$\vec{S}_v = \langle -u\sin(v), u\cos(v), 0 \rangle$$

$$S_u \times S_v = \det \begin{vmatrix} i & j & k \\ \cos(v) & \sin(v) & -2u \\ -u\sin(v) & u\cos(v) & 0 \end{vmatrix}$$

assume outward Positive orientation

$$= \langle 2u^2\cos(v), -(-2u^2\sin(v)), u\cos^2(v) + u\sin^2(v) \rangle$$

$$= u\langle 2u\cos(v), 2u\sin(v), 1 \rangle$$

Check $u = \frac{1}{2}, v = 0$, this is outward orientation.

$$\therefore \iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_D \langle u\sin(v), u\cos(v), 1-u^2\rangle \cdot u\langle 2u\cos(v), 2u\sin(v), 1\rangle \, dA$$

$$= \iint_{D_1} u\left(2u^2\sin(v)\cos(v) + 2u^2\sin(v)\cos(v) + (1-u^2)\right) dA$$

$$= \int_{u=0}^{1} u \int_{v=0}^{2\pi} \left(4u^2\cos(v)\sin(v) + (1-u^2)\right) dv\, du$$

$$\int \cos(v)\sin(v)\,dv = \frac{1}{2}\sin^2(v) + C$$

$$= \int_{u=0}^{1} u\left[2u^2\sin^2(v) + (1-u^2)v\right]_{v=0}^{2\pi} du$$

$$= \int_{u=0}^{1} u\left(0 + (1-u^2)(2\pi - 0)\right) du = 2\pi\int_{u=0}^{1} (u-u^3)\,du$$

$$= 2\pi\left[\frac{1}{2}u^2 - \frac{1}{4}u^4\right]_{u=0}^{1}$$

$$= 2\pi\left(\frac{1}{2} - \frac{1}{4} - 0\right) = \pi\left(1 - \frac{1}{2}\right) = \frac{\pi}{2}$$ ← This is all for S_1

we need to do S_2 now

$\vec{r}(u,v) = \langle u\cos(v), u\sin(v), 0\rangle$ on $D_2 = [0,1]\times[0,2\pi]$

$\vec{F}(r(u,v)) = \langle u\sin(v), u\cos(v), 0\rangle$

$\vec{r}_u = \langle \cos(v), \sin(v), 0\rangle$

$\vec{r}_v = \langle -u\sin(v), u\cos(v), 0\rangle$

$$\vec{r}_u \times \vec{r}_v = \det \begin{vmatrix} i & j & k \\ \cos(v) & \sin(v) & 0 \\ -u\sin(v) & u\cos(v) & 0 \end{vmatrix}$$

$$= \langle 0, 0, u\cos^2(v) + u\sin^2(v) \rangle = u\langle 0,0,1 \rangle$$

Note that this orientation is inward!, so we need to use $-\vec{r}_u \times \vec{r}_v$ instead!

$$\therefore \iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_{D_2} \vec{F}(\vec{r}(u,v)) \cdot -(\vec{r}_u \times \vec{r}_v)\, dA$$

$$= \iint_{D_2} \langle v\sin(v), u\cos(v), 0 \rangle \cdot -u\langle 0,0,1 \rangle\, dA$$

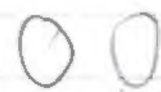
$$= \iint_{D_2} -u(0+0+0)\, dA = 0$$

$$\therefore \iint_{S} \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S} = \frac{\pi}{2} + 0 = \boxed{\frac{\pi}{2}}$$

16.8: Stokes's Theorem

Idea: want a version of green's therom which does not require the surface to sit flat in $z=0$ plane.

Therom (Stokes's Theorem): Let S be a surface in $\mathbb{R}^3$ which is piecewise — smooth and with ∂S a piecewise-smooth closed curve with one component. If $\vec{F}$ is a v.f. on $\mathbb{R}^3$ w/ continuous partial derivatives on S, then $\int_{\partial S} \vec{F} \cdot dr = \iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$



NB: we'll take this as a black box

Compute $\int_C \vec{F} \cdot d\vec{r}$ when $\vec{F} = \langle y^2, x, z^2 \rangle$ and C is the curve of intersection of the plane $y+z=2$ and cylinder $x^2+y^2=4$.

Sol: we need to use stokes's theorem, so we need $C = \partial S$ for surface S. lets use surface parametrized by $\vec{S}(r,\theta) = \langle r\cos\theta, r\sin\theta, 2 - r\sin\theta \rangle$ on $D = [0,4] \times [0,2\pi]$

Now by Stokes's theorem,

$$\int_C \vec{F}\cdot d\vec{r} = \int_{\partial S} \vec{F}\cdot dr = \iint_S \text{Curl}(\vec{F})\cdot d\vec{S} = \iint_D \text{Curl}(\vec{F}(S(r,\theta)))\cdot(S_r \times S_\theta)\, dA$$

$$\text{Curl}(\vec{F}) = \det \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ y^2 & x & z^2 \end{vmatrix}$$

$$\left(\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(x), -\left(\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial z}(y^2)\right), \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(y^2)\right.$$

$$= \langle 0, 0, 1 - 2y \rangle$$

$$\therefore \text{Curl}(\vec{F})(\vec{S}(r,\theta)) = \langle 0, 0, 1 - 2r\sin(\theta) \rangle$$

$$S_r = \langle \cos\theta, \sin\theta, \sin\theta \rangle$$

$$S_\theta = \langle -r\sin(\theta), r\cos(\theta), -r\cos(\theta) \rangle$$

$$S_r \times S_\theta = \det \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & -\sin\theta \\ -r\sin\theta & r\cos\theta & -r\cos\theta \end{vmatrix}$$

$$-r\sin\theta\cos\theta + r\sin\theta\cos\theta, -(-r\cos^2\theta - r\sin^2\theta), r\cos^2\theta + r\sin^2\theta$$

$$\langle 0, r, r \rangle$$

$\therefore \int_C \vec{F} \cdot d\vec{r} = \iint_D \langle 0, 0, 1 - 2r\sin\theta \rangle \cdot \langle 0, r, r \rangle \, dA$

$$\int_{r=0}^{1} \int_{\theta=0}^{2\pi} r(1 - 2r\sin\theta) \, d\theta \, dr$$

$$\int_{r=0}^{1} r \left[\theta + 2r\cos\theta\right]_0^{2\pi} dr$$

$$= (2\pi - 0) \int_{r=0}^{1} r \, dr = 2\pi \left(\tfrac{1}{2} \left[r^2\right]_0^1\right)$$

$$= \pi(1 - 0)$$

$$= \boxed{\pi}$$